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# GROUP PRESENTATIONS CORRESPONDING TO SPINES OF 3-MANIFOLDS. II

BY

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ABSTRACT. Let $\phi = \langle a_1, \ldots, a_n | R_1, \ldots, R_m \rangle$ denote a group presentation. Let $K_\phi$ denote the corresponding 2-complex. It is well known that every compact 3-manifold has a spine of the form $K_\phi$ for some $\phi$, but that not every $K_\phi$ is a spine of a compact 3-manifold. Neuwirth's algorithm (Proc. Cambridge Philos. Soc. **64** (1968), 603–613) decides whether $K_\phi$ can be a spine of a compact 3-manifold. However, it is impractical for presentations of moderate length.

In this paper a simple planar graph-like object, called a RR-system (railroad system), is defined. To each RR-system corresponds a whole family of compact orientable 3-manifolds with spines of the form $K_\phi$, where $\phi$ has a particular form (e.g., $\langle a, ba^mb^na^pb^n, a^mb^na^mb^q \rangle$), subject only to certain requirements of relative primeness of certain pairs of exponents. Conversely, every $K_\phi$ which is a spine of some compact orientable 3-manifold can be obtained in this way.

An equivalence relation on RR-systems is defined so that equivalent RR-systems determine the same family of manifolds. Results of Zieschang are applied to show that the simplest spine of 3-manifolds arises from a particularly simple kind of RR-system called a reduced RR-system.

Following Neuwirth, it is shown how to determine when a RR-system gives rise to a collection of closed 3-manifolds.

**1. Definitions and statements of preliminary results.** The authors wish to apologize for terminology which is, at times, too "cute". The reason for this terminology is fairly obvious in that these terms carry mnemonics for their definitions.

DEFINITION 1.1. Let $D$ be a regular hexagon in the plane, $E^2$. For each pair of opposite faces construct a finite set (possibly empty) of parallel line segments called *tracks* through $D$ with endpoints on these opposite faces. Call these sets of parallel segments *stations*. The hexagon $D$ together with the stations will be called a town. Let $\{D_i: i = 1, 2, \ldots, s\}$ be a set of disjoint twons in $E^2$. A *route* is an arc whose interior lies in $E^2 \sim \bigcup_{i=1}^s D_i$ which connects endpoints of tracks. A *trivial route* is a circle in $E^2 \sim \bigcup_{i=1}^s D_i$. A *RR-system* is the union in $E^2 \subset S^2$ of a finite set of disjoint towns and a

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finite set of disjoint routes in $S^2 \sim \bigcup_{i=i}^{s} D_i$ such that every route is either trivial or connects two endpoints of tracks, and each endpoint of every track intersects exactly one route in one of its endpoints. An example of a RR-system with two towns is shown in Figure 1.1 Two RR-systems are considered to be the "same" if there is a homeomorphism of $S^2$ onto itself which maps one RR-system onto the other. A track and a route are *adjacent* if they intersect. Adjacency generates an equivalence relation on the set of tracks and routes. We call the equivalence classes *companies*. Each *trivial route* by itself makes up a company called a *trivial company*.

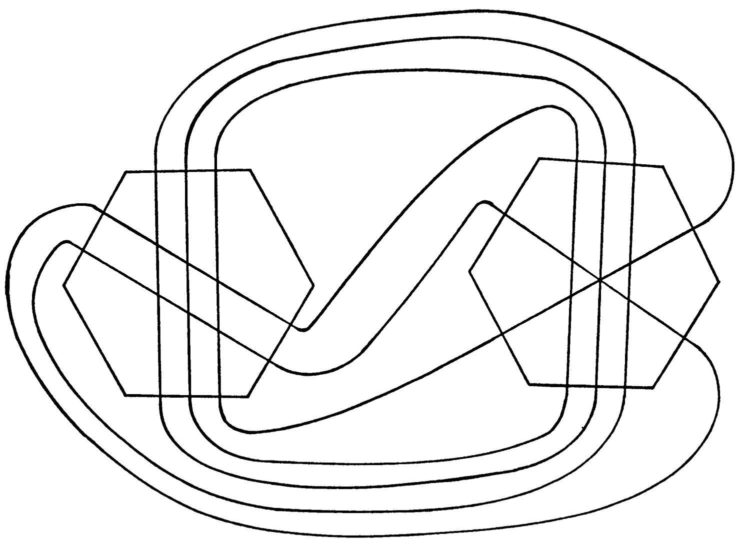


FIGURE 1.1

A RR-system gives rise to a family of presentations in a way that will be explained shortly. First we need some terminology for a general form that the presentations must have.

DEFINITION 1.2. Let $X = \{x_1, \ldots, x_k\}$ be a finite set called an alphabet. Let $H = \{\eta_1, \ldots, \eta_r\}$ be a finite set of indeterminates over the integers. Each element $x^\eta$ where $x \in X$ and $\eta \in H$ is called an *abstract syllable* in $X$ with base $x$ and exponent $\eta$. A finite sequence of abstract syllables is called an *abstract word*. The alphabet $X$ together with a finite set $W$ of (not necessarily distinct and possibly empty) abstract words or abstract relators is called an abstract presentation, which we denote by $\Phi = \langle X | W \rangle$. If $\Phi$ is an abstract presentation with exponents in $\{\eta_1, \ldots, \eta_r\}$ we may regard $\Phi$ as a function whose domain is the $r$-fold cartesian product of the integers and whose range is a set of finite group presentations. This function is defined by replacing each indeterminate in the abstract relators of $\Phi$ by its corresponding integer value. Denote the presentation obtained by substituting $n_i$ for $\eta_i$ by $\Phi(n_1, n_2, \ldots, n_r)$. For example, if $\Phi = \langle x_1, x_2 | x_1^{\eta_2} x_2^{\eta_1} x_1^{\eta_1} x_1^{\eta_2} \rangle$ then $\Phi(-3, 5) = \langle x_1, x_2 | x_1^5 x_2^{-3} x_2^{-3} x_1^5 \rangle$.

Given a RR-system $R$ one can derive a corresponding abstract presentation as follows. The generators of the abstract presentation will be in one-to-one correspondence with the towns of $R$. We think of these generators $x_1, \ldots, x_k$ as being names of the towns. In each town $x_i$ we start at some vertex of the boundary of its hexagon and proceed clockwise (according to an orientation of $S^2$) along an edge. This edge corresponds to a station called $m_i$. Orient the tracks of this station so that the positive direction is toward this edge. Label the stations corresponding to the second and third edges encountered by $m_i + p_i$ and $P_i$, respectively, and orient the tracks of these stations toward the respective edges. Beginning at some point on some route we "ride the company train" (going straight ahead on the track as we pass through a town) until we return to the starting point. As we enter each station the station master stamps our ticket with the name of the town, the name of the station, and a "minus" sign if our direction of travel opposes the orientation of the track. A typical stamp looks like $x_2^{m_2}$ or $x_4^{-m_4}$. Each stamp is placed to the right of the previous stamp. When we have completed our ride our ticket has stamped on it a sequence of symbols that makes up an abstract word in the alphabet $\{x_1, \ldots, x_k\}$. If a company is trivial then the corresponding abstract word is empty and we denote it by 1. The alphabet $\{x_1, \ldots, x_k\}$ together with the set of abstract words (or *abstract relators*) obtained above (one for each company) is an abstract presentation obtained from the RR-system $R$.

In this construction there were several choices which influence the abstract presentation obtained. They were

(1) labelling the towns,

(2) the orientation of $S^2$,

(3) at which vertex of the hexagon we start in labelling the stations of each town,

(4) on which route of the company we begin each of our train rides, and

(5) which way each train goes around its company.

We define an equivalence relation on the abstract presentations obtained from RR-systems so that the resulting equivalence classes are uniquely determined by the RR-systems. This equivalence relation is generated by the following (which correspond, respectively, to the above choices):

(1) relabelling the generators,

(2) interchanging $m_i$ and $p_i$ for all $i$,

(3) for some $i$ replace $m_i$ by $-p_i$, $m_i + p_i$ by $m_i$, and $p_i$ by $m_i + p_i$ in each

abstract relator,

(4) cylically conjugate an abstract relator, and

(5) take the inverse of an abstract relator.

Suppose that, for some $i$, $D_i$ has an empty station $m_i + p_i$. Let $D_i'$ be a town whose stations, labeled clockwise are $m_i$, $m_i + p_i$, and $p_i$, have the same respective numbers of tracks as the stations $m_i$, $p_i$, and $m_i + p_i$ of $D_i$ again labeled clockwise. Then any RR-system having $D_i$ as a town is topologically equivalent with the RR-system obtained by replacing $D_i$ with $D_i'$. For this reason we add the following to our generating set of the above equivalence relations:

(6) If $m_i + p_i$ does not appear as an exponent of $a_i$ in the abstract presentation, then leave $m_i$ fixed and interchange $m_i + p_i$ and $p_i$.

DEFINITION 1.3. It will be convenient for stating some theorems to associate with each town in a RR-system the starting point for labeling the stations in a town. A RR-system together with a set of starting points (one for each town) will be called a *pointed RR-system*. We shall always assume that the stations in the $i$th town are labeled clockwise from the starting point by $m_i$, $m_i + p_i$, and $p_i$.

DEFINITION 1.4. A group presentation is an alphabet $X$ together with a collection of not necessarily distinct elements from the free semigroup of $X$ and its formal inverse $X^{-1}$. Thus $\langle x|xx^{-1}x\rangle$ is a different presentation from $\langle x|x\rangle$ which is different from $\langle x|x, 1\rangle$. Two presentations will be considered to be the same if the 2-complexes determined by them are isomorphic. Thus $\langle x|xx^{-1}x\rangle$ is considered to be the same as $\langle x|xxx^{-1}\rangle$.

THEOREM 1.4. *Let $R$ be a RR-system with $\Phi$ an abstract presentation obtained from $R$. Let $\phi$ be a group presentation obtained from $\Phi$ by choosing integer values $m_i^*$ and $p_i^*$ for $m_i$ and $p_i$, respectively, so that $(m_i^*, p_i^*) = 1$; $i = 1, \ldots, S$. Then the 2-complex $K_\phi$ corresponding to $\phi$ is a spine of an orientable 3-manifold.*

DEFINITION 1.5. Let $R$ be a RR-system and let $\Phi$ be an abstract presentation obtained from $R$. Let $\phi$ be a group presentation obtained from $\Phi$ by choosing integer values $m_i^*$ and $p_i^*$ for $m_i$ and $p_i$, respectively, so that $(m_i^*, p_i^*) = 1$; $i = 1, 2, \ldots, s$. Then we will say that $\phi$ *originates* from $R$, $\phi$ *originates from* $\Phi$ and $\Phi$ *originates from* $R$. We denote by $\Omega_R$ the family of all group presentations originating from $R$ and we denote by $\mathfrak{M}_R$ the collection of all orientable compact 3-manifolds having spine $K_\phi$ for some $\phi \in \Omega_R$.

In subsequent work we will often not distinguish between abstract presentations in which exponents are indeterminates and group presentations in which the exponents are integers, it being left to the reader to determine from the context which is intended.

The converse of Theorem 1.4 also holds.

THEOREM 1.6. *Let* $\phi$ *be a group presentation such that* $K_\phi$ *is a spine of an orientable* 3-*manifold. Then there exists a RR-system R from which* $\phi$ *originates.*

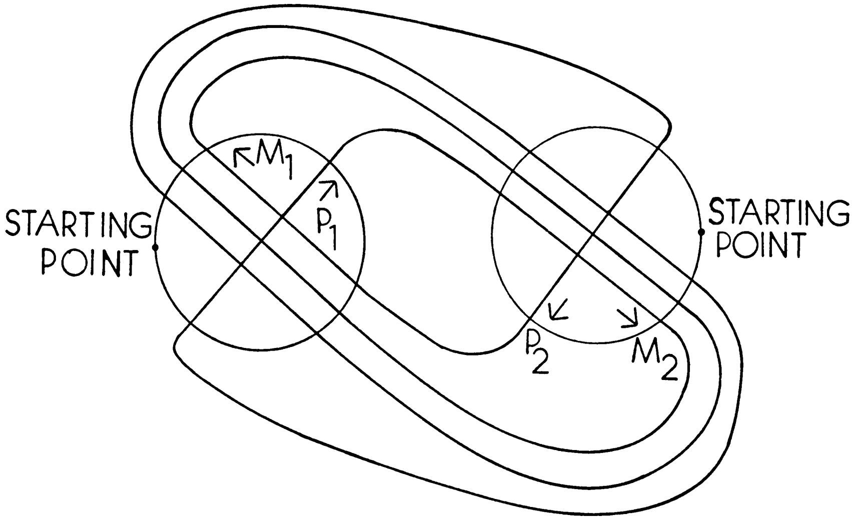



FIGURE 1.2

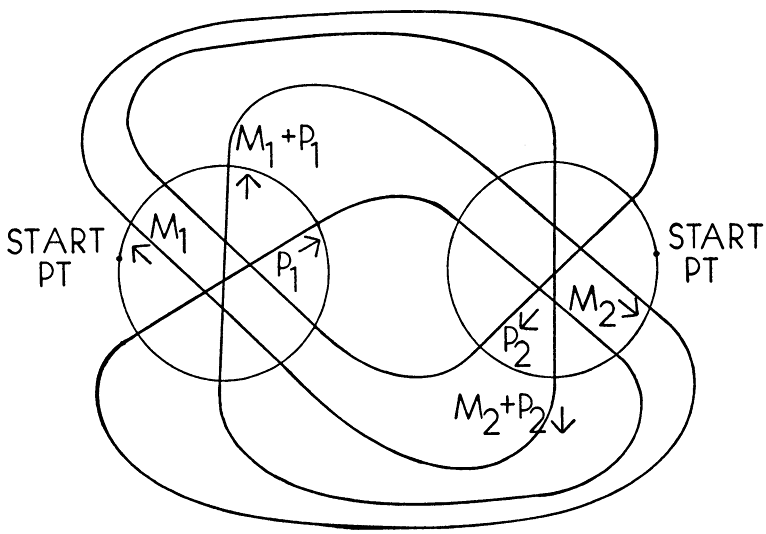



FIGURE 1.3

One might ask if the same presentaiton can originate from different RR-systems. The answer is yes. For example Figures 1.2 and 1.3 show

RR-systems whose abstract presentations are, respectively,

$$\langle x_1, x_2 | x_1^{m_1} x_2^{m_2} x_1^{m_1} x_2^{p_2}, x_1^{m_1} x_2^{m_2} x_1^{p_1} x_2^{m_2} \rangle \quad \text{and}$$

$$\langle x_1, x_2 | x_1^{m_1} x_2^{m_2+p_2} x_1^{m_1} x_2^{p_2}, x_1^{m_1+p_1} x_2^{m_2} x_1^{p_1} x_2^{m_2} \rangle.$$

If we let $m_1 = m_2 = 1$ and $p_1 = p_2 = 2$ in the first abstract presentation we get $\langle x_1, x_2 | x_1 x_1 x_1 x_2^2, x_1 x_2 x_1^2 x_2 \rangle$, whereas substitution of $m_1 = m_2 = p_1 = p_2 = 1$ in the second gives $\langle x_1, x_2 | x_1 x_2^2 x_1 x_2, x_1^2 x_2 x_1 x_2 \rangle$. Actually the RR-systems carry more information than this; and, in fact, the manifolds determined are not the same. The first being the lens space $L(5, 2)$ while the second is $L(5, 1)$.

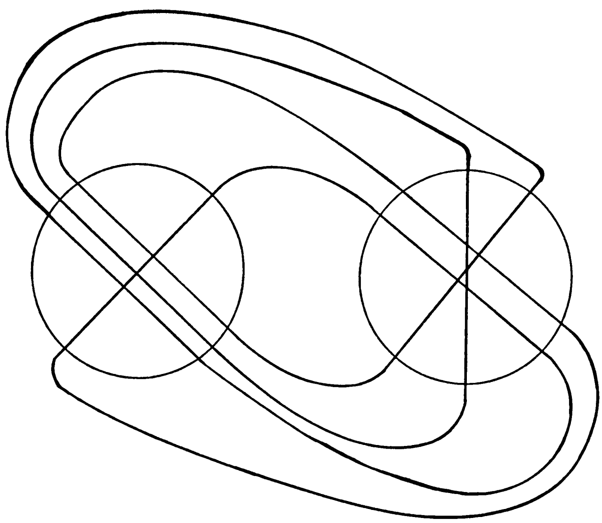


FIGURE 1.4

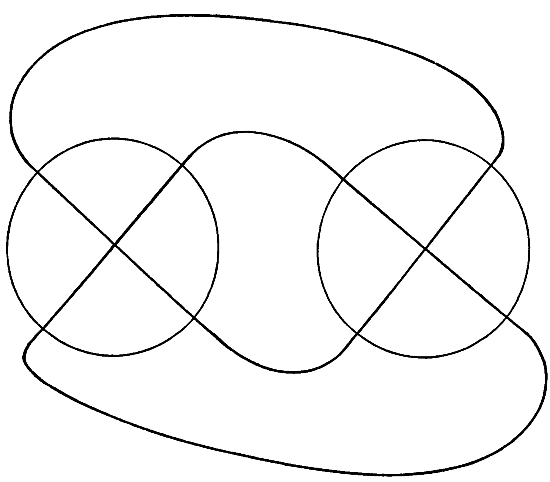


FIGURE 1.5

A more unsettling situation occurs because two different systems can determine exactly the same family of manifolds, e.g. Figures 1.4 and 1.5 show such RR-systems. Their respective presentations are

$$\langle x_1, x_2 | x_1^{m_1}x_2^{m_2+p_2}, x_1^{m_1}x_2^{m_2}x_1^{p_1}x_2^{m_2}x_1^{m_1}x_2^{p_2}\rangle \quad \text{and}$$

$$\langle x_1, x_2 | x_1^{m_1}x_2^{m_2}, x_1^{p_1}x_2^{p_2}\rangle.$$

Using techniques developed in §3, we can easily show that any manifold with a spine corresponding with the first abstract presentation also has a spine of the second type. In fact all these manifolds must be lens spaces (see [8] for a proof that manifolds with spines whose presentations have the second form are lens spaces).

2. **Equivalence of RR-systems.** We now give procedures for producing a new RR-system $\tilde{R}$ from a given RR-system $R$ and show that $R$ and $\tilde{R}$ correspond to the same family of manifolds, i.e., $\mathfrak{M}_R = \mathfrak{M}_{\tilde{R}}$. We will do this by defining operations on RR-systems that are analogous to those given in the Free Cancellation and Multiplication Theorems of §3.

DEFINITION 2.1. If $R$ is a RR-system and $r$ is a route whose endpoints lie on one town and are not endpoints of the same track, $r$ is called a *cancellation route*. A RR-system is *reduced* if it has no cancellation routes.

The following lemma is an immediate consequence of our routine for obtaining an abstract presentation from a RR-system.

LEMMA 2.2. *There is a one-to-one correspondence between cancellation routes in a RR-system $R$ and pairs of adjacent syllables with the same base in an abstract presentation originating from $R$.*

Suppose $r$ is a cancellation route in $R$ with endpoints in the town $D_i$. Further suppose that one of the complementary domains of $S^2 \sim (D_i \cup r)$ does not intersect $R$. (We take $R$ to denote the union of all its tracks and routes.) We must have

(1) the endpoints of $r$ lie in the same station, or

(2) they lie in different stations.

In case (1) we have an adjacent pair of syllables of the form $x_i^k x_i^{-k}$ occurring in a relator $W$. These correspond to two tracks $t_1$ and $t_2$ in the same station, and $W$ is of the form $a^k a^{-k} W'$ where $W'$ is a (possibly empty) abstract word. Let $r_1$ and $r_2$ denote the routes intersecting $t_1 \sim r$ and $t_2 \sim r$, respectively. Let $s$ denote the straight line segment on the boundary of $D_i$ between the ends of $t_1$ and $t_2$ not intersecting $r$. Let $N_s$ be a regular neighborhood of $s$ in $S^2 \sim \mathring{D}_i$. Let $s'$ be the arc in Bd $N_s \sim D_i$ which connects $r_1$ and $r_2$ (see Figure 2.1). Replace the routes $r$, $r_1$, $r_2$ and the tracks $t_1$ and $t_2$ by the single route $((r_1 \cup r_2) \sim N_s) \cup s'$. This operation replaces $W$ by $W'$ in the corresponding abstract presentation.

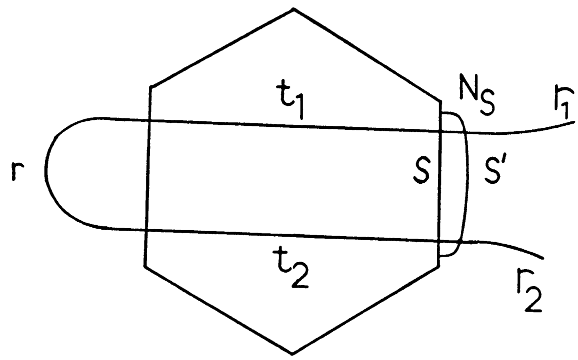



FIGURE 2.1

In case (2) we have an adjacent pair of syllables of the form $a_i^{k_1}a_i^{k_2}$ where $|k_1|$ and $|k_2|$ are different elements of $\{m_i, m_i + p_i, p_i\}$ (by the absolute value sign we mean "remove the minus sign if it is there"). These syllables correspond to tracks $t_1$ and $t_2$ in different stations. Let $t'$ be a line segment in $D_i$ joining the endpoints of $t_1$ and $t_2$ that do not intersect $r$. Replace the tracks $t_1$ and $t_2$ and the route $r$ by the single track $t'$. This operation replaces $a_i^{k_1}a_i^{k_2}$ by $a_i^{k_1+k_2}$ (see Figure 2.2).

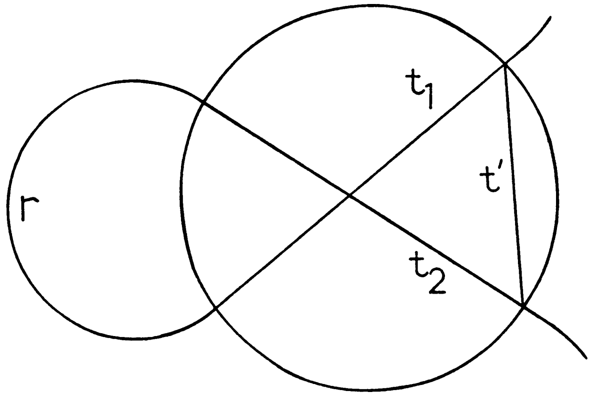



FIGURE 2.2

If $\tilde{R}$ denotes a RR-system obtained from $R$ in either case (1) or (2), we will say that $\tilde{R}$ was obtained from $R$ by *cancellation*.

If $R$ was a pointed RR-system then $\tilde{R}$ can also be assumed to be pointed with the chosen point in each town remaining invariant in the above construction. One should not assume that the starting point precedes the first track of the first station in the clockwise orientation, only that the starting point identifies the head of the first station.

THEOREM 2.3. RR-CANCELLATION. *Let $R$ be a pointed RR-system and let $\Phi$*

*be an abstract presentation corresponding to R. Let $\tilde{R}$ be obtained from R by cancellation, and let $\tilde{\Phi}$ be the corresponding abstract presentation obtained from $\Phi$. Suppose that $\phi$ and $\tilde{\phi}$ originate from $\Phi$ and $\tilde{\Phi}$, respectively, by the same choice of integer values for the $m_i$'s and $p_i$'s. Then the 2-complexes $K_\phi$ and $K_{\tilde{\phi}}$ are spines of the same orientable 3-manifold.*

Let $C_1$ and $C_2$ be distinct companies in $R$ such that some route $r_1$ of $C_1$ can be connected to a route $r_2$ of $C_2$ by an arc $t$ whose interior lies in $S^2 - (R \cup \bigcup_{i=1}^n D_i)$. See Figure 2.3 for the following construction. We

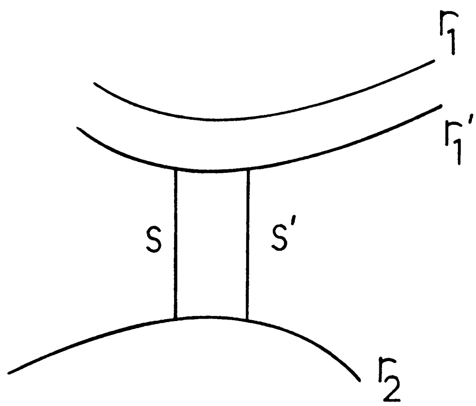



FIGURE 2.3

construct a new company $C_1'$ which follows "parallel" to the tracks and routes of $C_1$. Furthermore we construct $C_1'$ so that it separates $r_1$ from $r_2$ (i.e., so that it intersects $t$). Using $C_1'$ and $C_2$ we construct a new company as follows. Let $r_1'$ be the route in $C_1'$ that duplicates $r_1$. Connect $r_1'$ and $r_2$ by two "parallel" arcs $s$ and $s'$ whose interiors are in $S^2 - (R \cup C_1' \cup (\bigcup_{i=1}^n D_i))$ (we may take $s$ to be the appropriate subarc of $t$). Delete from $r_1'$ and $r_2$ the respective subarcs between $s$ and $s'$. Thus we obtain a new closed curve which is really a company that we denote by $C_1 *_s C_2$. The new RR-system $\tilde{R}$ is $R$ with $C_2$ replaced by $C_2 *_s C_2$. Again a pointed $R$ gives rise to a pointed $\tilde{R}$.

We construct two abstract words $W_1'$ and $W_2$ that correspond to the companies $C_1'$ and $C_2$. To get $W_1'$ we begin at $s \cap r_1'$ and proceed around $C_1'$ in such a way that we travel all tracks in $C_1$ before reaching $s' \cap r_1'$. To get $W_2$ we start at $s' \cap r_2$ and proceed around $C_2$ covering all tracks of $C_2$ before reaching $s \cap r_2$. If $\tilde{\Phi}$ is obtained from $\Phi$ by replacing $W_2$ with $W_1' W_2$ then $\tilde{\Phi}$ is an abstract presentation of $\tilde{R}$. We will say that $\tilde{R}$ is obtained from $R$ by *multiplication*. The following theorem follows immediately from the multiplication theorem of §3.

THEOREM 2.4 (RR-MULTIPLICATION). *Suppose that R is a pointed RR-system* *and that $\tilde{R}$ is obtained from $R$ by multiplication. Let $\Phi$ and $\tilde{\Phi}$ be abstract presentations corresponding, respectively, to $R$ and $\tilde{R}$ as described above. Further, suppose that $\phi$ and $\tilde{\phi}$ are group presentations which originate from $\Phi$ and $\tilde{\Phi}$ by choosing the same values for the $m_i$'s and $p_i$'s. Then $K_\phi$ and $K_{\tilde{\phi}}$ are spines of the same orientable 3-manifold.*

The operations of cancellation and multiplication generate an equivalence relation on the collection of RR-systems. The following corollary follows immediately from Theorems 2.3 and 2.4.

COROLLARY 2.5. *If $R$ and $\tilde{R}$ are equivalent RR-systems then $\mathfrak{M}_R = \mathfrak{M}_{\tilde{R}}$.*

THEOREM 2.6 (RR-ELIMINATION). *Let $R$ be an RR-system with abstract presentation $\Phi$. Suppose that $\Phi$ has an abstract relator $W$ which defines an abstract syllable, i.e. $W = x_i^{-k}U$ where $k$ is the name of a station in the town corresponding to $x_i$ and $U$ is an abstract word not involving $x_i^{\pm k}$. Let $\Phi'$ be the abstract presentation obtained from $\Phi$ by substituting $U$ for each occurrence of the syllable $x^k$ except the occurrence in $W$. Then there is an equivalent RR-system $R'$ such that $\Phi'$ is an abstract presentation corresponding to $R'$.*

There is still another way of transforming one RR-system into another without changing the family of manifolds determined. This transformation arises from Zieschang's work [11], [12] and corresponds algebraically to a Nielsen transformation of the generators in the group presentation. We shall first describe how the RR-system is to be changed, then review briefly the required results of Zieschang, and finally in §5 prove that these transformations do not change the manifolds determined.

DEFINITION 2.7. Let $R$ be a RR-system and let $x_0$ be a town in $R$. Construct a new track $t$ in $x_0$. Let $T$ be a "fattening" of $t$ in $x_0$ so that $T$ does not

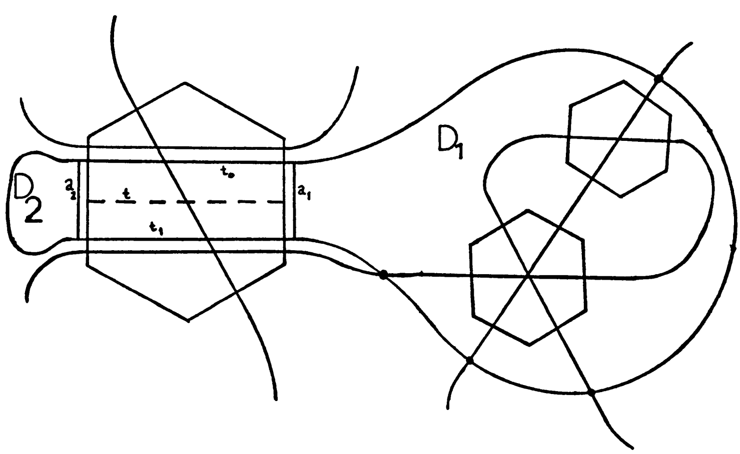



FIGURE 2.4

intersect any of the tracks parallel with $t$. More specifically, one can construct $T$ by choosing two tracks $t_0$ and $t_1$ parallel with $t$ on each side of $t$ and not containing between them any track other than $t$. (See Figure 2.4.) Let $D_1$ be a disk such that $\operatorname{Int} D_1 \sqcap x_0 = \varnothing$, $\partial D_1 \sqcap x_0$ is a component $a_1$ of $T \sqcap \partial x_0$ and $(\partial D_1 \sim x_0) \sqcap R$ is a finite set of crossing points on routes outside the other towns of $R$. Let $D_2$ be a disk such that $\operatorname{Int} D_2 \sqcap x_0 = \varnothing$, $\partial D_2 \sqcap x_0 = a_2$ (the other component of $T \cap \partial x_0$) and $(D_2 \cap R) \sim x_0 = \varnothing$. The new RR-system $\tilde{R}$ is to agree with $R$ except in $D_1 \cup D_2 \cup T$. Let $f\colon D_1 \to D_2$ be an orientation preserving homeomorphism such that $f(\overline{\partial D_1 \sim a_1}) = a_2$. For all points of $(\partial D_1 \sim a_1) \cap R$ we construct disjoint arcs in $D_1 \cup T$, each connecting a point of $(\partial D_1 \sim a_1) \cap R$ with its image under $f$. $\tilde{R} \cap (D_1 \cup D_2 \cup T)$ is defined to be the union of $f(R \cap D_1)$ and these arcs. Also included in $\tilde{R}$ are any segments of original tracks that cross $T$. We describe this process by saying that *the towns enclosed in $D_1$ have been dragged through the town $x_0$ along the track $t$*. The general process will be referred to as a *handle dragging transformation*.

Suppose $D_1$ has the following additional properties. (1) $\operatorname{Int} D_1$ contains only the town $x_{i_0}$ and no other town. (2) The number of points of intersection of $\partial D_1$ with a route is the number of endpoints of that route lying on $x_{i_0}$. Think of $D_1$ as a "fattening" of $x_{i_0}$ together with a thin tube connecting this fattening with $x_0$ as required above. If we now perform a handle drag with respect to a disk $D_1$ with these special properties, we say that the town $x_{i_0}$ has been dragged through the town $x_0$ along the track $t$.

Of course this operation might also lead to the creation of some cancellation routes. If no towns are "caught" inside a resulting cancellation route, one can perform the cancellation without changing the manifolds determined (Theorem 2.3).

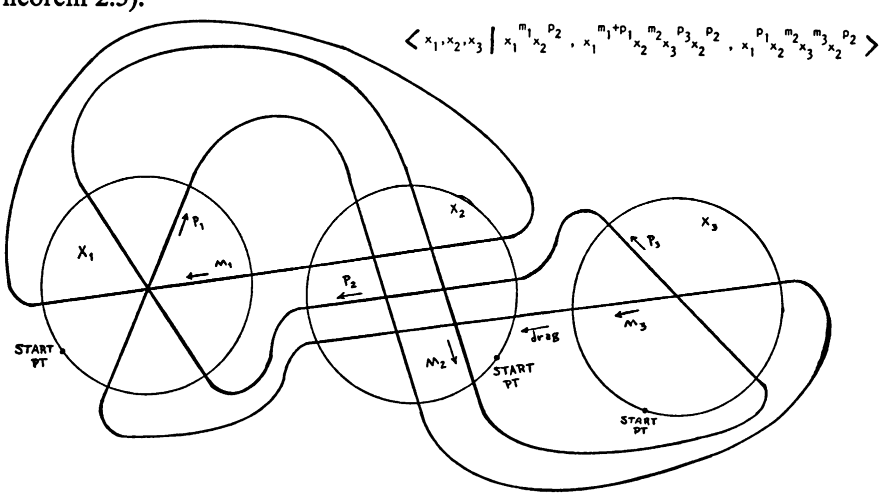
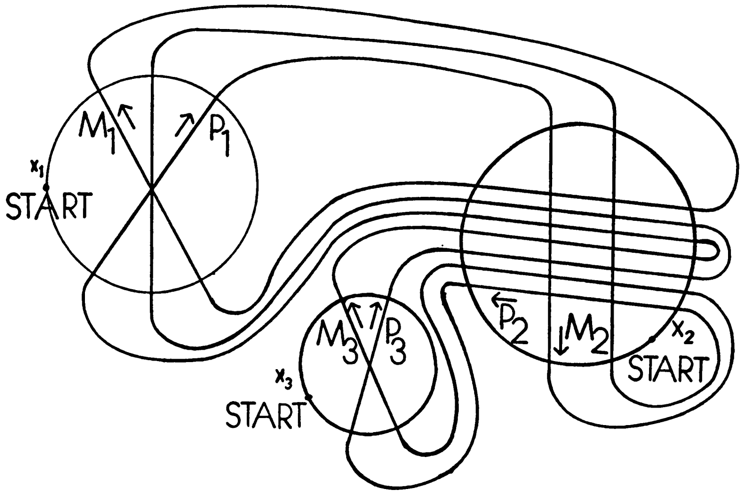



FIGURE 2.5a




replace $x_3$ by $x_2^{P_2} x_3 x_2^{-P_2}$

FIGURE 2.5b

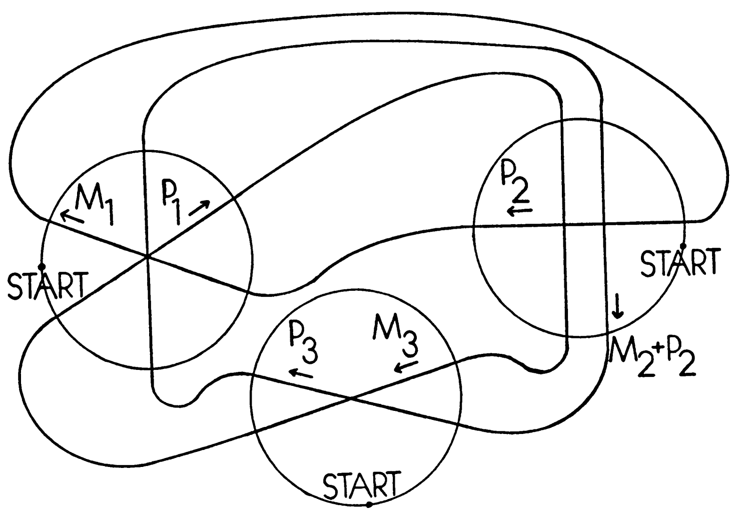



FIGURE 2.5c

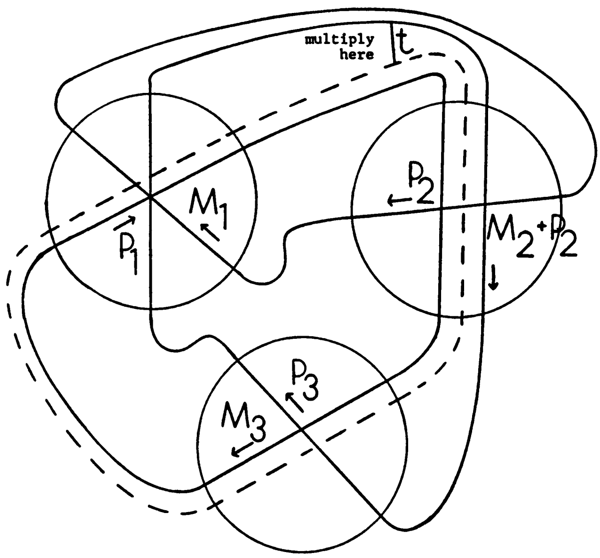



FIGURE 2.5d

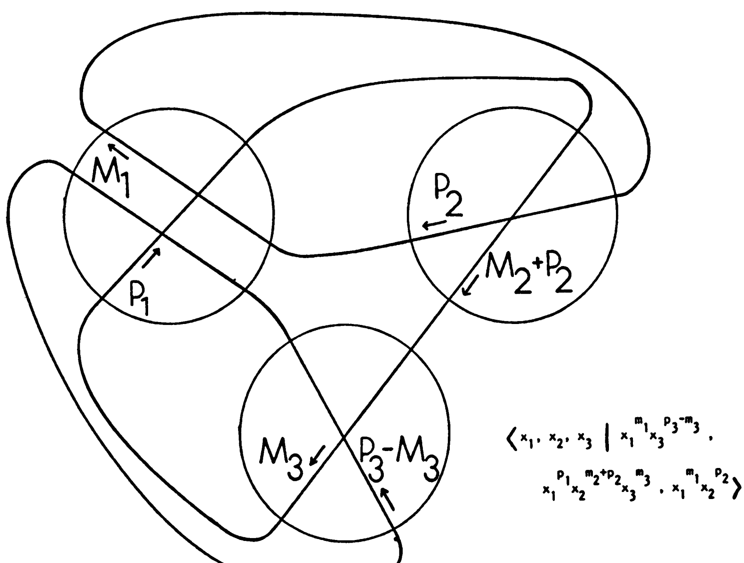



FIGURE 2.5e

Two RR-systems will be called equivalent if there is a finite sequence of RR-multiplications, RR-cancellations, and handle dragging transformations leading from one to the other.

Figures 2.5(a), (b), (c), (d), (e) show a sequence of such transformations which simplify the RR-system in (a) to that shown in (e). (a) shows the initial RR-system and indicates a handle drag through the middle town, (b) indicates the result of this transformation, (c) shows the same system after the cancellations have been performed, (d) shows one of the companies copied for RR-multiplication (dotted) and (e) shows the result after multiplication and cancellation.

THEOREM 2.8. *Let $R'$ be a RR-system obtained from $R$ by dragging the $x_i$ town through the town $x_0$ along the track $m_0$. Let $\Phi_R$ be an abstract presentation corresponding to $R$ and let $\Phi'$ be obtained from $\Phi_R$ by replacing $x_i^{n_i}$ by $x_0^{-m_0}x_i^{n_i}x_0^{m_0}$ for each abstract syllable $x^{n_i}$ with base $x_i$. Then $\Phi'$ is an abstract presentation corresponding to $R'$ and for every evaluation of the indeterminates $m_i$ and $p_i$ we have $\Phi(\{m_i, p_i\})$ and $\Phi'(\{m_i, p_i\})$ correspond to spines of the same 3-manifold.*

Before beginning the process of proving our statements we shall state what we shall call the fundamental theorem for RR-systems, which is:

THEOREM 2.9. *Let $M$ be a compact 3-manifold and let $\phi$ be a presentation corresponding to a spine $K_\phi$ of $M$. There is a reduced RR-system with corresponding abstract presentation $\Phi_R$ and there is a choice of the exponents $m_i$ and $p_i$ in $\Phi_R$ with the following properties*:

(a) *$\Phi_R(\{m_i, p_i\})$ corresponds to a spine of $M$*;

(b) *$\Phi_R(\{m_i, p_i\})$ has total length not greater than $\phi$.*

*In addition it follows from the fact that $R$ is a reduced RR-system that*

(c) *$\Phi_R(\{m_i, p_i\})$ is freely reduced.*

The power of the above theorem is in the guarantee that in studying reduced RR-systems we are not complicating the types of spines to be studied. In addition, of course, the equivalence relation induced by multiplication, cancellation, and dragging towns through towns enables one to get infinite collections of presentations all of which determine the same manifold. We shall develop this idea more fully in subsequent sections.

3. **Extended $P$-graphs.** It has become apparent that we need to consider 2-complexes having one or more 2-cells whose entire boundaries are attached to the vertex. In [7] such 2-complexes are excluded from consideration for the reason that the $P$-graph (the boundary of a regular neighborhood of the vertex) would not be a graph, since the 2-cells mentioned above would give rise to circles with no vertices. This being a minor difficulty we provide this

extension and briefly discuss some consequences.

Theorem 3.3 gives a topological parallel to the Tietze Transformation of elimination and deserves some attention. It is a direct consequence of Theorems 3.1 and 3.2 which are really the change-of-spine theorems of [7] revisited after the above extension. Further, Theorem 3.3 has the advantage of not requiring any more than the fact that $K_\phi$ be a spine of some orientable 3-manifold.

*Notation.* If $\phi$ is a presentation we denote by $K_\phi$ the 2-complex corresponding to $\phi$ (see [7]). Every presentation $\phi$ determines a unique $P$-graph $P_\phi$ obtained as the boundary of a regular neighborhood of the vertex in $K_\phi$. If $\phi = \langle x_1, \ldots, x_n | W_1, \ldots, W_k \rangle$ then the points at which the oriented 1-cell in $K_\phi$ corresponding to $x_i$ intersects $P_\phi$ are called *endpoints* of $P_\phi$ and are denoted by $x_i^-$ and $x_i^+$, where following the 1-cell from $x_i^-$ to $x_i^+$ not passing through the vertex of $K_\phi$ is the "positive direction" in the 1-cell. If a relator $W_j$ is empty (written $W_j = 1$) then corresponding to $W_j$ in $P_\phi$ is a 1-sphere.

Now select regular neighborhoods of $x_i^+$ and $x_i^-$ in $R_\phi$. The points called *vertices* on the boundary of these neighborhoods will be denoted, respectively, by $x_{i,j}^+$; $j = 1, 2, \ldots, k_i$, and $x_{i,j}^-, j = 1, 2, \ldots, k_i$. Furthermore we label our points so that $B(x_{i,j}^+) = x_{i,j}^-$ where $B$ is the handle correspondence as defined in [5], [7] and [8]. If there is an arc between $x_{i,j}^{\varepsilon_1}$, $\varepsilon_1 = +$ or $-$, and $x_{k,l}^{\varepsilon_2}$, $\varepsilon_2 = +$ or $-$, that does not pass through any endpoint, then we say that $x_{i,j}^{\varepsilon_1}$ is *connected to* $x_{k,l}^{\varepsilon_2}$ and write $A(x_{i,j}^{\varepsilon_1}) = x_{k,l}^{\varepsilon_2}$. Note that $A^2 = 1$. Suppose now that $P_\phi$ is embedded in $S^2$. Then this embedding determines a cyclic ordering for the vertices $\{x_{i,j}^{\varepsilon}, j = 1, 2, \ldots, k_i\}$ about $x_i$. Denote by $C(x_{i,j}^{\varepsilon})$ the first vertex encountered going clockwise (according to some fixed orientation) around $x_i^{\varepsilon}$ from $x_{i,j}^{\varepsilon}$. If $BC = C^{-1}B$ then we say that $P_\phi$ is *faithfully embedded* in $S^2$. If $P_\phi$ is faithfully embedded in $S^2$ then this embedding uniquely determines an orientable compact 3-manifold which has $K_\phi$ for a spine (see [8]). This manifold is called *the manifold corresponding to the embedded P-graph* or more simply just *the corresponding manifold.* Of course not all $P$-graphs have faithful embeddings and, in fact, most do not. For example, the $P$-graph of $\phi = \langle x_1, x_2 | x_1^2 x_2^2, x_1^4, x_2^6 \rangle$ cannot be faithfully embedded in $S^2$. Thus $K_\phi$ is not a spine of an orientable 3-manifold.

*The length of a presentation* is the sum of the lengths of its relators. The *length of a P-graph* is the length of the corresponding presentation. A *syllable* of a word in the free group $F(x_1, \ldots, x_n)$ is a maximal subword containing only one generator called the *base* for the syllable. Let $\phi = \langle x_1, \ldots, x_n | W_1, \ldots, W_k \rangle$ be a presentation with the property that the base for the first and last syllables of $W_i$ are different unless $W_i$ contains at most one syllable. The total number of syllables in $W_1, \ldots, W_k$ is called the syllable length of $\phi$.

Using our slightly extended definition of $P$-graph it is easy to see that Theorems 2.1 and 2.2 of [8] are still valid. We give these theorems here.

THEOREM 3.1 (FREE CANCELLATION). *Let $\phi$ be a group presentation whose $P$-graph $P_\phi$ is faithfully embedded in $S^2$. Let $W = xx^{-1}W'$ be a relator of $\phi$. Let $\lambda$ denote the loop at $x^+$ corresponding to the cancellation pair and suppose that $S^2 \sim \lambda$ has a component $D$ such that $D \cap P_\phi = \varnothing$. If $\phi'$ denotes the group presentation obtained from $\phi$ by replacing $W$ with $W'$ then $P_{\phi'}$ has a faithful embedding in $S^2$. Moreover $K_\phi$ is a spine of the orientable 3-manifold $M^3$ if and only if $K_{\phi'}$ is a spine of $M^3$.*

Note that this theorem is a generalization of its eariler version since we no longer require that $W'$ be nonempty.

If the loop at $x^+$ (or $x^-$) satisfies the conditions of Theorem 3.1 we say that *the corresponding free cancellation can be performed.*

Let $\phi = \langle x_1, x_2, \ldots, x_n | W_1, W_2, \ldots, W_k \rangle$ be a group presentation. For each $i = 1, 2, \ldots, k$, let $e_i$ and $e_i'$ be distinct points of $P_\phi$ such that the following hold:

(1) The points $e_i$ and $e_i'$ lie on the interior of the edge of $P_\phi$ corresponding to the space between the last and first letters of $W_i$ (provided $W_i$ is not empty). If $W_i$ is empty the $e_i$ and $e_i'$ lie on the 1-sphere corresponding to $W_i$.

(2) One starts at $e_i$ and traces through that part of $P_\phi$ corresponding to $W_i$ before passing $e_i'$ and returning to $e_i$.

THEOREM 3.2 (MULTIPLICATION). *Let $\phi$ be a group presentation with at least two relators and suppose that $P_\phi$ is faithfully embedded in $S^2$. Let $\alpha$ and $\beta$ be two "parallel" arcs whose interiors lie in $S^2 - P_\phi$ such that $\alpha$ connects $e_1$ and $e_2'$ and $\beta$ connects $e_2$ and $e_1'$. If $\phi'$ is obtained from $\phi$ by replacing the relator $W_1$ with $W_1W_2$, then $K_{\phi'}$ has a faithful embedding in $S^2$. Further, $K_\phi$ is a spine of the orientable 3-manifold $M^3$ if and only if $K_{\phi'}$ is.*

EXAMPLE. Consider the presentation $\phi = \langle a | 1 \rangle$. $P_\phi$ has two different faithful embeddings in $S^2$ as shown in Figures 3.1 and 3.2. The embedding in Figure 3.1 corresponds to $K_\phi$ being a spine of $S^2 \times S^1$, and that in Figure 3.2 corresponds to a solid torus with a "bubble" in it.

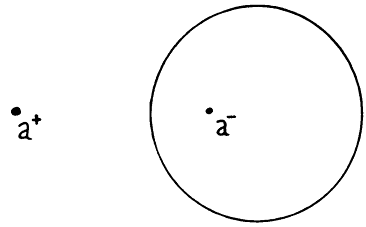



FIGURE 3.1

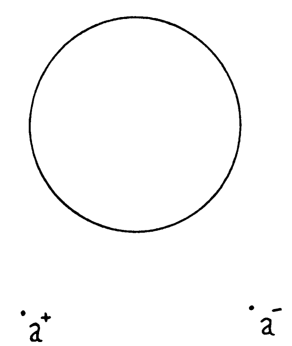



FIGURE 3.2

We now prove a change-of-spine theorem which corresponds to the Tietze Transformation of elimination.

THEOREM 3.3 (ELIMINATION). *Suppose that $\phi$ is a group presentation and that $K_\phi$ is a spine of an orientable 3-manifold $M^3$. Suppose also that one of the relators of $\phi$ is a defining relator for one of the generators, say $W = x^{-1}U$ where $U$ is a word not involving $x$. Let $\phi'$ be obtained from $\phi$ by the indicated elimination (i.e., the generator $x$ and the relator $W$ are eliminated, and in all other relators each occurrence of $x$ is replaced with $U$). Then $K_{\phi'}$ is a spine of $M^3$.*

PROOF. We use the Multiplication and Free Cancellation Theorems. The argument goes by induction on the total number $k$ of occurrences of $x$ in the relators of $\phi$. If $k = 1$ then the only occurrence of $x$ is in $W = x^{-1}U$. In this case we merely collapse the 2-cell corresponding to $W$ across the 1-cell corresponding to $x$, this being a free face.

Suppose then that $k > 1$. Then in $P_\phi$ corresponding to the relator $W$ there is just one vertex $x_0^+$ "near" the endpoint $x^+$. Let $x_1^+ = C(x_0^+)$. Then $x_1^+$ lies on an edge of $P_\phi$ which corresponds to a relator $W_1 \neq W$. Write $W_1 = Vx$. Apply the Multiplication Theorem to replace $W_1$ by $W_1' = WW_1 = Vxx^{-1}U$. Then apply the Free Cancellation Theorem to replace $W_1'$ with $W_1'' = VU$. The presentation $\phi''$ so obtained is $\phi$ with $W_1$ replaced with $W_1'' = VU$. So $x$ occurs in the relators of $\phi''$ only $k - 1$ times. Further, $K_{\phi''}$ is also a spine of $K_\phi$. This completes the induction and the theorem is proved.

## 4. Proofs of Theorems 1.4, 1.6, 2.3 and 2.6.

PROOF OF THEOREM 1.4. We will obtain a faithful embedding of the $P$-graph of $\phi$ in $S^2$ from the RR-system $R$ by replacing the towns with appropriate faithfully embedded syllable graphs. (See [7].) Note that $\phi$ can be obtained from any abstract presentation that is equivalent to $\Phi$ by appropriate choices of values for $m_i$ and $p_i$, $i = 1, \ldots, s$. In particular, we will construct an abstract presentation $\Phi'$ equivalent to $\Phi$ such that the appropriate choices for $m_i$ and $p_i$ are all nonnegative. Let $D_i$ be a town of $R$ and let $V_{i,0}$ be the starting point (which was used to label the stations of $D_i$ in constructing $\Phi$).

For each $i = 1, \ldots, s$, label the vertices of the hexagon $D_i$ clockwise from $V_{i,0}$ by $V_{i,1}, V_{i,2}, \ldots, V_{i,s}$. Construct $\Phi'$ by using as starting points $V_{i,r_i}$, $i = 1, \ldots, s$, where

$$r_i = \begin{cases} 0 & \text{if } m_i^* \geqslant 0 \text{ and } p_i^* \geqslant 0, \\ 1 & \text{if } m_i^* < 0, p_i^* \geqslant 0, \text{ and } m_i^* + p_i^* \geqslant 0, \\ 2 & \text{if } m_i^* < 0, p_i^* \geqslant 0, \text{ and } m_i^* + p_i^* < 0, \\ 3 & \text{if } m_i^* < 0 \text{ and } p_i^* < 0, \\ 4 & \text{if } m_i^* \geqslant 0, p_i^* < 0, \text{ and } m_i^* + p_i^* < 0, \\ 5 & \text{if } m_i^* \geqslant 0, p_i^* < 0, \text{ and } m_i^* + p_i^* \geqslant 0. \end{cases}$$

Let $m_i'$ and $p_i'$ be the names of the stations used for $\Phi'$. Then $\phi$ may be obtained from $\Phi'$ by choosing $m_i'$ and $p_i'$ to have the respective values $m_i'^*$ and $p_i'^*$ where

$$(m_i'^*, p_i'^*) = \begin{cases} (m_i^*, p_i^*) & \text{if } r_i = 0, \\ (m_i^* + p_i^*, -m_i^*) & \text{if } r_i = 1, \\ (p_i^*, -(m_i^* + p_i)) & \text{if } r_i = 2, \\ (-m_i^*, -p_i^*) & \text{if } r_i = 3, \\ (-(m_i^* + p_i^*), m_i^*) & \text{if } r_i = 4, \\ (-p_i^*, m_i^* + p_i^*) & \text{if } r_i = 5. \end{cases}$$

Note that $m_i'^*$ and $p_i'^*$ are nonnegative integers and that $(m_i'^*, p_i'^*) = 1$. For each town $D_i$ we construct a faithfully embedded syllable graph with syllable exponents $m_i'^*$, $m_i'^* + p_i'^*$, and $p_i'^*$ so that there are the same respective numbers of syllables with these exponents as there are tracks in the respective stations $m_i'$, $m_i' + p_i'$, and $p_i'$ (see Theorem 4.4 of [6]). This construction gives a faithful embedding of the $P$-graph corresponding to $\phi$.

Note that it is possible that $m_i^* = 0$ (in which case $p_i^*$ must be $+ 1$). A route could connect both endpoints of the same track in the station $m_i$. If this happens then we would have a trivial relator in $\phi$. There are two other ways in which $\phi$ could have a trivial relator. One, of course, would arise from the RR-system $R$ having a trivial company. Or we could choose 0 for the value of every exponent appearing in an abstract relator, provided this is done consistently with $(m_i^*, p_i^*) = 1$, $i = 1, \ldots, s$.

Proof of Theorem 1.6. Since $K_\phi$ is a spine of an orientable 3-manifold we know that $P_\phi$ has a faithful embedding in $S^2$. For each generator $x_i$ of $\phi$ construct an arc $t_i$ connecting $x_i^-$ to $x_i^+$ so that

(1) $t_i \cap t_j = \emptyset$ whenever $i \neq j$, and

(2) if $\lambda$ denotes any edge of $P_\phi$ then $\lambda^0$ intersects each $t_i$ in at most one point which is a crossing point.

For each $t_i$ let $D_i$ be a regular neighborhood (with respect to the RR-system) of $t_i$ in $S^2$. Then $D_i \cap P_\phi$ is a faithfully embedded syllable graph with syllables in $x_i$ and exponents 1 and 0. Moreover, $P_\phi \sim \bigcup D_i$ is a disjoint union of arcs. Hence we may replace each disc $D_i$ with an appropriate town to form an RR-system $R$ from which $\phi$ originates.

Proof of Theorem 2.3. Suppose that the adjacent syllable pair where the cancellation takes place is $x_i^{k_1}x_i^{k_2}$, i.e., $x_i^{k_1}x_i^{k_2}$ is replaced by $x_i^{k_1+k_2}$ if $k_1$ and $k_2$ represent different abstract exponents, or is deleted if $d_1 = -k_2$ as abstract exponents. Suppose that $k_1^*$ and $k_2^*$ are the integers that are substituted for $k_1$ and $k_2$, respectively, in obtaining $\phi$ and $\tilde{\phi}$. We consider two cases.

*Case* (a). $k_1^*$ and $k_2^*$ are of the same sign or one of them is 0. There is nothing to prove since $P_\phi$ and $P_{\tilde{\phi}}$ are identical $P$-graphs and have the same faithful embedding.

*Case* (b). $k_1^*$ and $k_2^*$ are both nonzero and have opposite sign. We apply the Free Cancellation Theorem 3.1 as many times as necessary (namely $\min(|k_1^*|, |k_2^*|)$ times) to replace the syllable pair $x_i^{k_1^*}\, x_i^{k_2^*}$ with $x_i^{k_1^*+k_2^*}$.

Proof of Theorem 2.6. The proof proceeds by induction on the number of tracks in the station $k$ in town $x_i$. If there is only one such track there is nothing to prove. If there are 2 or more tracks in this station, then we copy the company corresponding to $W$ and multiply that company with a company one of whose tracks is next to the track corresponding to $x_i^k$ in $W$. Furthermore this multiplication is to be performed along an arc close to the town $x_i$. After this multiplication has been performed a RR-Cancellation can be done yielding a new RR-system with fewer tracks in station $k$.

5. **Zieschang's work as related to $P$-graphs.** In this section we shall apply Zieschang's work in [8] and [12] to $P$-graphs and show how to establish the results on dragging a town through another town (Theorem 2.8). Using these results we shall also establish the fundamental theorem for RR-systems (Theorem 2.9).

Definition 5.1. Let $\phi = \langle x_1, x_2, \ldots, x_n | R_1, R_2, \ldots, R_m \rangle$ be a group presentation. Let

$$\tilde{\phi} = \langle x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_n | \tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_m \rangle$$

be the presentation obtained from $\phi$ by the substitution of $\tilde{x}_i^{\varepsilon_1} x_j^{\varepsilon_2}$ for $x_i$, $i \neq j$, in each of the words $R_1, R_2, \ldots, R_m$, where $\varepsilon_1$ and $\varepsilon_2$ are $\pm 1$. Then we say that $\tilde{\phi}$ is obtained from $\phi$ by an *elementary Nielsen transformation*. The result of a finite sequence of such transformations will be called a *Nielsen transformation of* $\phi$.

Note that neither $\phi$ nor $\tilde{\phi}$ is required to be freely reduced and that no assumption about performing free cancellations is made. For example, transforming $\langle x_1, x_2 | x_1x_2x_1x_2, x_1^2x_2x_1x_2^{-2}\rangle$ to

$$\langle \tilde{x}_1, x_2 | \tilde{x}_1x_2^{-1}x_2\tilde{x}_1x_2^{-1}x_2, \tilde{x}_1x_2^{-1}\tilde{x}_1x_2^{-1}x_2\tilde{x}_1x_2^{-3}\rangle$$

is an elementary Nielsen transformation obtained by substituting $\tilde{x}_1x_2^{-1}$ for $x_1$.

DEFINITION 5.2. Let $\phi = \langle x_1, \ldots, x_n | W_1, \ldots, W_k\rangle$, let $P_\phi$ be faithfully embedded in $S^2$ and let $D^2$ be a disk in $S^2$ with the following properties: $x_1^\varepsilon \in \operatorname{Int} D^2$ and $x_1^{-\varepsilon} \in S^2 \sim D^2$ where $\varepsilon$ is $+$ or $-$, $\partial D^2$ intersects each edge of $P_\phi$ in isolated crossing points and $\partial D^2$ does not intersect any endpoint of $P_\phi$. Furthermore we think of the regular neighborhoods of $x_i^+$ and $x_i^-$, $i = 1, 2, \ldots, n$, which define $x_{i,r}^\pm$ as being small enough so that they lie entirely in $\operatorname{Int} D^2$ or in $S^2 \sim D^2$. Denote by $D_\varepsilon^2$ and $D_{-\varepsilon}^2$ the (small) regular neighborhoods of $x_1^\varepsilon$ and $x_1^{-\varepsilon}$, respectively, which determine the vertices $\{x_{1,r}^\varepsilon: r = 1, \ldots, k_1\}$ and $\{x_{1,r}^{-\varepsilon}: r = 1, \ldots, k_1\}$, i.e. $\partial D_\varepsilon^2$ and $\partial D_{-\varepsilon}^2$ contain $\{x_{1,r}^\varepsilon: r = 1, \ldots, k_1\}$ and $\{x_{1,r}^{-\varepsilon}: r = 1, \ldots, k_1\}$, respectively. Let $F^2$ be a small regular neighborhood of $x_1^{-\varepsilon}$ in $\operatorname{Int} D_{-\varepsilon}^2$. (See Figure 5.1.) Delete that part of $P_\phi$ in $\operatorname{Int} D_{-\varepsilon}^2$ from $P_\phi$. Let $h$: $D^2 \sim D_\varepsilon^2 \to D_{-\varepsilon}^2 \sim F^2$ be an orientation reversing homeomorphism such that $h(x_{1,r}^\varepsilon) = B(x_{1,r}^\varepsilon) = x_{1,r}^{-\varepsilon}$ for each $r = 1, 2, \ldots, k_1$. The intersection of the new $P$-graph $P_\phi^{\sim}$ with $D_{-\varepsilon}^2$ −

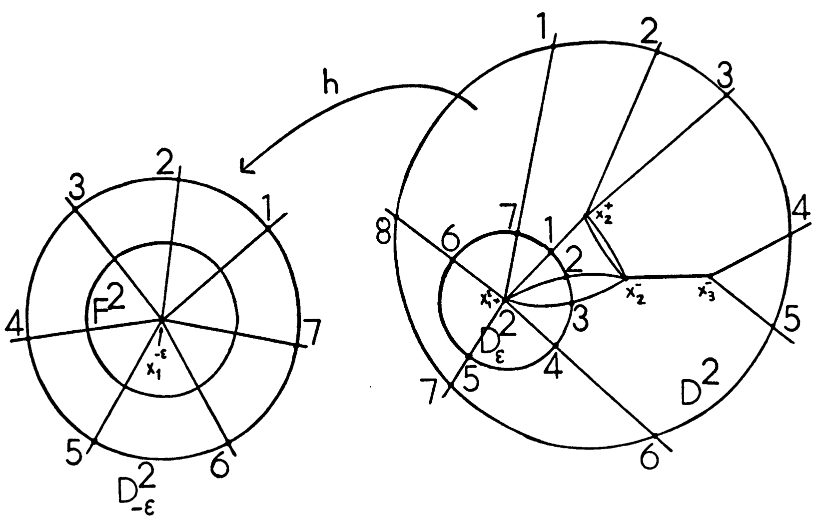



FIGURE 5.1

$F^2$ is to be $h(D^2 - D^2_\varepsilon \cap P_\phi)$. $P_{\tilde\phi} \cap F^2$ is to be the join of the center $\tilde x_1^{-\varepsilon}$ of $F^2$ with the points of $P_{\tilde\phi} \cap \partial F^2$. $P_{\tilde\phi} \cap D^2$ is obtained by taking the join of the center $\tilde x_1^\varepsilon$ of $D^2$ with $\partial D^2 \cap P_{\tilde\phi}$. In $S^2 \sim (D^2 \cup D^2_{-\varepsilon})$, $P_\phi$ and $P_{\tilde\phi}$ coincide. The vertices in $P_{\tilde\phi}$ at $\tilde x_1^\varepsilon$ and $\tilde x_1^{-\varepsilon}$ are taken to be $P_\phi \cap \partial D^2 = \{\tilde x^\varepsilon_{1,r}: r = 1, 2, \dots, \tilde k_1\}$ and $h(P_\phi \cap \partial D^2) = \{\tilde x^{-\varepsilon}_{1,r}: r = 1, 2, \dots, \tilde k_1\}$, and $B$ is defined on these vertices to agree with $h$. (Figure 5.2 shows $P_{\tilde\phi}$ for the situation depicted

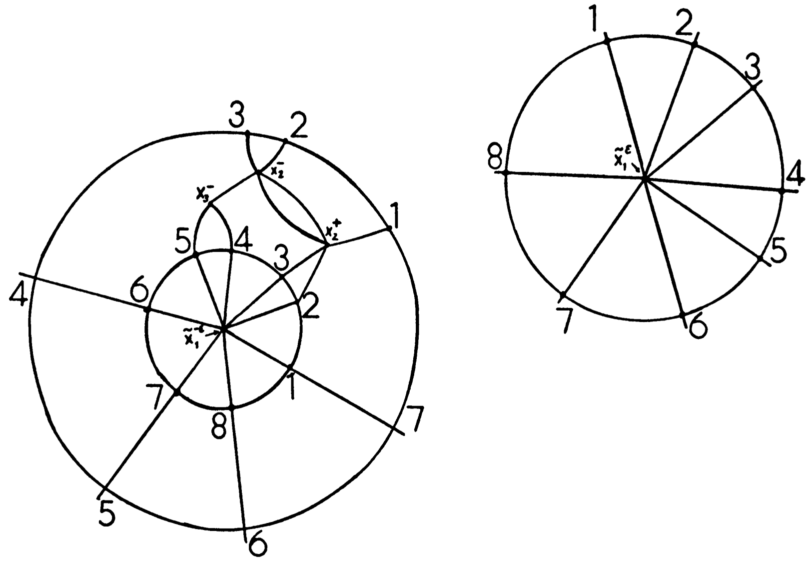



FIGURE 5.2

in Figure 5.1.) Elsewhere $B$ is defined for $P_{\tilde\phi}$ exactly as it was for $P_\phi$. If $N$ and $\tilde N$ denote the lengths of $\phi$ and $\tilde\phi$, respectively, then $\tilde N = N - k_1 + \tilde k_1$. We shall say that $P_{\tilde\phi}$ *was obtained from* $P_\phi$ *by dragging the material in* $D^2$ *over the* $x_1^\varepsilon$ *handle.*

Algebraically $\tilde\phi$ is obtained from $\phi$ by a Nielsen transformation and insertion or deletion of free cancellations of the form $\tilde x_1 \tilde x_1^{-1}$ or $\tilde x_1^{-1}\tilde x_1$. The Nielsen transformation is defined as follows. If $x_i^\partial \in D^2$ ($\partial$ is $+$ or $-$) and $x_i^{-\partial} \notin D^2$, then $\tilde x_i^\partial = x_i^\partial x_1^{-\varepsilon}$; if $x_i^+$ and $x_i^-$ are in $D^2$ then $\tilde x_i = x_1^\varepsilon x_i x_1^{-\varepsilon}$; if $x_i^+$ and $x_i^-$ are not in $D^2$ then $\tilde x_i = x_i$.

DEFINITION 5.3. Using the above notation if $P_\phi$ is faithfully embedded and the free cancellations introduced by the transformation from $P_\phi$ to $P_{\tilde\phi}$ can be performed, we shall say that the corresponding Nielsen transformation *freely reduces*. The transformation obtained by a Nielsen transformation that freely reduces followed by these free cancellations on either the $P$-graph or its presentation will be called a *reduced Nielsen transformation on the P-graph* or *its presentation*.

DEFINITION 5.4. If $\tilde{N} < N$ we call $D^2$ a *simplifying disk for the embedded* $P_\phi$.

Given a free reduced faithfully embedded $P$-graph, we may examine it to see if it is possible to shorten the corresponding presentation. For each pair of endpoints $x_i^+$ and $x_i^-$ we look for a disk $D^2$ whose boundary intersects $P_\phi$ in fewer points than the number of edges at $x_i^+$. If such a disk exists the corresponding freely reduced Nielsen transformation shortens the length of $\phi$. If no simplifying disk exists Zieschang [12] has shown that $\phi$ is Nielsen reduced, i.e. that no freely reduced Nielsen transformation of $\phi$ will shorten its length.

If there is more than one point of intersection of $\partial D^2$ with an edge, each additional pair of such points leads to a free cancellation of the type $x_1^\varepsilon x_1^{-\varepsilon}$ or $x_1^{-\varepsilon} x_1^\varepsilon$ depending on whether the part of the edge between this pair of points lies outside or inside $D^2$.

The above observations lead us to the following.

THEOREM 5. *Let* $\phi = \langle x_1, \ldots, x_n | R_1, \ldots, R_k \rangle$ *where* $R_1 = R_1' x_1^\varepsilon x_1^{-\varepsilon}$ *and* $P_\phi$ *is faithfully embedded in* $S^2$. *The loop in* $P_\phi$ *corresponding to the above free cancellation separates* $S^2$. *Let* $D^2$ *be a regular neighborhood of the complementary domain not containing* $x_1^{-\varepsilon}$. *Then there is a faithfully embedded P-graph* $P_{\tilde{\phi}}$ *in* $S^2$ *with the following properties*:

(i) $P_{\tilde{\phi}}$ *is obtained from* $P_\phi$ *by dragging the material in* $D^2$ *over the* $x_1^\varepsilon$ *handle.*

(ii) *The length of* $\tilde{\phi}$ *is less than the length of* $\phi$.

(iii) *The length of each relator of* $\tilde{\phi}$ *is not greater than the length of the corresponding relator in* $\phi$.

Let us now discuss the relationship of Zieschang's work in [11] and [12] to $P$-graphs. Given a faithfully embedded $P$-graph $P_\phi$ one can construct a simple system of curves on the boundary of a handlebody (Vollbrezeln) as follows. We think of $P_\phi$ as being embedded in $S^2 = \partial B^3$. For each pair of endpoints $x_i^+$ and $x_i^-$ we attach a 1-handle to $B^3$ on regular neighborhoods of $x_i^+$ and $x_i^-$ so as to get an orientable handle on $B^3$. Now draw disjoint arcs on the boundary of this handle connecting $x_{i,j}^+$ to $x_{i,j}^-$. The result is what Zieschang calls a simple system of curves on the boundary of the handlebody. This can be viewed in another way. If we take an embedding of $K_\phi$ in a 3-manifold $M^3$ and consider the intersection of $K_\phi$ with a regular neighborhood of the 1-skelton of $K_\phi$ we get an equivalent simple system of curves. This observation and construction is due to Neuwirth [5]. In both constructions the curves determined are not unique but can vary according to how many times they circle the handles (see Figure 5.3.) The nonuniqueness of these induced curves does not change the manifold determined by attaching 2-handles along these curves [5]. In fact this is one of the principal advantages

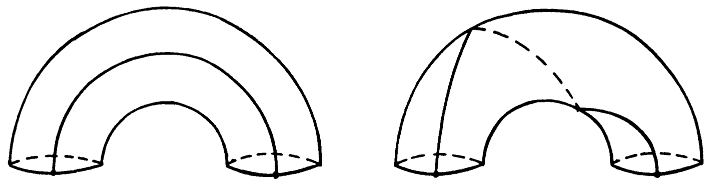


FIGURE 5.3

of $P$-graphs over studying simple closed curves on handlebodies, that is $P$-graphs ignore the twisting.

If we take this handlebody with its system of curves and cut it by a complete system of meridian disks (vollständige Zerschneidung) obtaining a cube $B^3$ with paired disks on its boundary corresponding to the meridian disks, we get exactly a $P$-graph with regular neighborhoods of its endpoints replaced by disks. Now Zieschang defined an operation which replaces a meridian disk by another meridian disk.

This can be done by selecting a disk $D$ in $B^3$ with boundary on $\partial B^3$ and interior in Int $B^3$ so that $\partial D$ does not intersect any of the paired disks in $\partial B^3$ and so that $\partial D$ separates some pair $D_i^+$ and $D_i^-$ of these disks. Now we reidentify $D_i^+$ and $D_i^-$ and cut the resulting handlebody along $D$. The resulting cube with holes (Löcher) gives rise to a new $P$-graph. It is not difficult to see that the resulting $P$-graph is exactly that obtained by dragging the material in a component of $S^2 \sim \partial D$ over the $x_i$ handle. It follows that the manifold determined by this $P$-graph $P_{\tilde{\phi}}$ is the same as that determined by $P_\phi$.

THEOREM 5.6. *If $P_\phi$ is faithfully embedded in $S^2$ and $P_{\tilde{\phi}}$ is obtained from $P_\phi$ by dragging the material in $D^2$ over the $x_i^\varepsilon$ handle, then the faithfully embedded $P_{\tilde{\phi}}$ determines the same manifold as the faithful embedding of $P_\phi$.*

THEOREM 5.7 (ZIESCHANG). *If $P_\phi$ is faithfully embedded in $S^2$, then there is a reduced Neilsen transformation that decreases the length of $\phi$ if and only if there is a reducing disk for $P_\phi$.*

In the above theorem we do not require that the Nielsen transformation reducing the length be a transformation that can be performed on the $P$-graph. What this theorem says is that if there is a Nielsen transformation that reduces the length of $\phi$, then there is a (possibly different) Nielsen transformation that shortens the length and can be performed on the $P$-graph without changing the manifold determined. We would like to mention without proof that this theorem has been proved from an entirely different point of view. One can characterize this proof algebraically as a finite sequence of two kinds of transformations of the corresponding presentations.

The first is introduction of a new generator $x_0$ and replacing a relator, say $UV$, by two relators $Ux_0^1$ and $x_0^{-1}V$. This corresponds to subdividing the disk corresponding to the relator $UV$. The second is multiplication. We illustrate this with the following example. Let $\phi = \langle a, b | aba^2b, abab^2ab^2 \rangle$. There is a faithful embedding of $P_\phi$ in $S^2$. Its RR-system is shown in Figure 5.4.

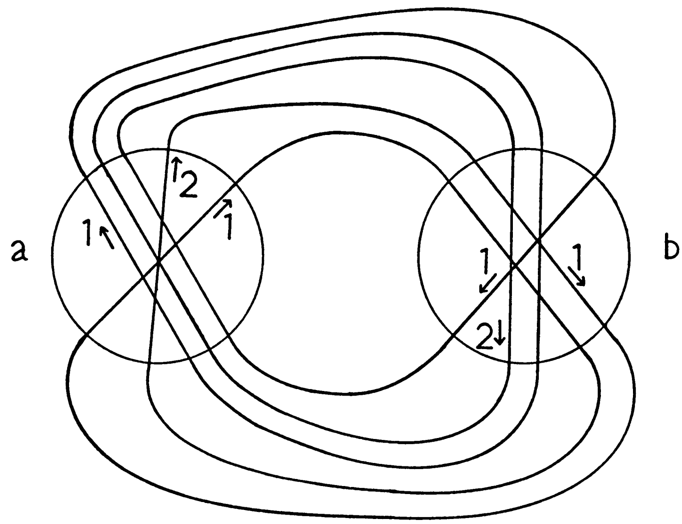



FIGURE 5.4

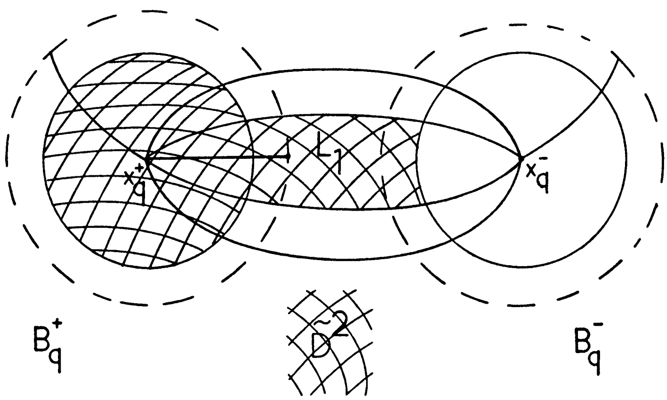
We introduce a new generator $c$ and replace the first relator with the relators $abc$ and $c^{-1}a^2b$. The first relator is a defining relator for $a$. If we eliminate $a$ using this relator and perform the resulting free cancellations we get the presentation $\langle b, c | c^{-3}b^{-1}, c^{-2}bc^{-1}b \rangle$. The first relator is a defining relator for $b$. Eliminating $b$ we get $\langle c | c^{-2}c^{-3}c^{-1}c^{-3} \rangle = \langle c | c^9 \rangle$ which corresponds to a spine of a lens space. Note that by subdivision and elimination (which follows immediately from the multiplication theorem) we were able to accomplish the Nielsen transformation of substituting $c^{-1}$ for $ab$.

CONJECTURE. The Nielsen transformations of $\phi$ which can be performed on $P_\phi$ by a dragging material over handles are exactly those that arise from subdivision and elimination.

**6. Proof of the Fundamental Theorem.** Let $\phi'$ be a presentation whose $P$-graph $P_{\phi'}$ is faithfully embedded in $S^2$. By dragging the material in disks over handles we arrive at a presentation $\phi$ whose length cannot be reduced by a reduced Nielsen transformation (Theorem 5.6). Denote by $P_\phi(x_q)$ the subgraph of $P_\phi$ consisting of all edges between $x_q^+$ and $x_q^-$ plus all line segments joining $x_q^-$ to a vertex $x_{qj}^-$ or joining $x_q^+$ to $x_{qj}^+$ for $j = 1, 2, \ldots, k_q$.

Here the set $\{x_{q,j}^+: j = 1, 2, \ldots, k_q\}$ is assumed to be the complete set of vertices at $x_q^+$. In general, $S^2 \sim P_\phi(x_q)$ has several components. We define $P$-contiguity and $P$-component for these regions just as for $P$-graphs (Definition 7.1). It may happen that there is more than one $P$-component of $S^2 \sim P_\phi(x_q)$.

Let us assume for the moment that there is only one $P$-component of $S^2 \sim P_\phi(x_q)$. Let us designate the (topological) components of $S^2 \sim P_\phi(x_q)$ by $L_1, L_2, \ldots, L_\lambda$. We assume that the subscripts $1, 2, \ldots, \lambda$ have been chosen so that $L_i$ is $P$-contiguous over the $x_q^+$ handle with $L_{i+1}$ for $i = 1, 2, \ldots, \lambda - 1$. We are now ready to modify $P_\phi$ to get a new embedding $P_{\phi'}$ that is equivalent with that of $P_{\phi'}$. Let $B_q^+$ and $B_q^-$ be regular neighborhoods of $x_q^+$ and $x_q^-$, respectively, in $S^2$. Let $\tilde{D}^2 = B_q^+ \cup \bar{L}_1 \sim \operatorname{Int} B_q^-$ and let $D^2$ be "concentric" with $\tilde{D}^2$ and slightly smaller. (See Figure 6.1.) Now we drag




$\tilde{D}^2 = B_q^+ \cup \bar{L}_1 \sim \operatorname{Int} B_q^-$

FIGURE 6.1

the material in $D^2$ over the $x_q^+$ handle to get a new $P$-graph $P_{\phi_1}$ with corresponding presentation $\phi_1$. Note that since $P_\phi$ is Nielsen reduced the number of points of intersection of $P$ with $D^2$ is the same as the number of edges at $x_q^+$. It follows that $P_{\phi_1}$ has the same length as $P_\phi$. Note also that if all vertices of $P_\phi$ lie in preferred components of $P_\phi(x_r)$, $r \neq q$, then the same property applies to the image of $P_\phi(x_r)$ after being dragged over the $x_q^+$ handle. This operation moves any part of $P_\phi$ lying in $L_1$ into $L_2$. After $\lambda - 1$ operations of this type we arrive at a $P$-graph with no vertices lying in any $L_i$, $i \neq \lambda$. If there is more than one $P$-component of $S^2 \sim P_\phi(x_q)$, say $u$ of them, we connect these $P$-components by an arc in $S^2$ which intersects $P_\phi(x_q)$ at exactly $u - 1$ points. This is possible since such a separation can only occur from the presence of two or more single syllable relators of the form $x_q^{m_q}$. In this case the number of $P$-components of $S^2 \sim P_\phi(x_q)$ is one less than the number of such relators and the syllables that are not themselves relators do not influence the number of $P$-components of $S^2 \sim P_\phi(x_q)$. We call the components intersected by this arc preferred components. (See [7] and [8] for an analysis of the embeddings of syllable graphs.) We now perform the above handledragging operation in each $P$-component until all vertices lie in preferred components of $S^2 \sim P_\phi(x_q)$. Performing the above operations successively for each of the generators, we arrive at a faithfully embedded $P$-graph $P_{\tilde{\phi}}$ with the following properties: (a) $\tilde{\phi}$ has the same length as $\phi$; (b) $P_{\tilde{\phi}}$ is obtained from $P_\phi$ by pulling material over handles; (c) for each generator $x_q$ of $\tilde{\phi}$ there is a collection of preferred regions of $S^2 \sim P_{\tilde{\phi}}(x_q)$ so that all vertices of $P_{\tilde{\phi}}$ except $x_q^+$ and $x_q^-$ lie in a preferred region. Note also that $\phi$ and $\tilde{\phi}$ have the same length.

Suppose $\tilde{\phi} = \langle x_1, \ldots, x_n | W_1, \ldots, W_k \rangle$. Let us now relabel the pairs of endpoints of $P_{\tilde{\phi}}$ so that $x_q^+$ and $x_q^-$ lie on the boundary of the same component of $S^2 \sim P_{\tilde{\phi}}$ if $1 \leqslant q \leqslant s$ and $x_q^+$ and $x_q^-$ do not lie on the boundary of the same component of $S^2 \sim P_{\tilde{\phi}}$ if $s < q \leqslant n$. For each $q$ with $1 \leqslant q < s$ we choose a disk $D_q$ in $S^2$ with the following properties: $x_q^+$ and $x_q^-$ lie in Int $D_q$ and $x_r^+$ and $x_r^-$ lie in $S^2 \sim D_q$ for $r \neq q$; $D_q \cap D_r = \varnothing$ for $q \neq r$, $1 \leqslant q \leqslant s$, $1 \leqslant r \leqslant s$; $\partial D_q$ intersects each edge of $P_{\tilde{\phi}}$ with exactly one endpoint in $D_q$ exactly once; Int $D_q$ contains all edges of $P_{\tilde{\phi}}$ that connect $x_q^+$ and $x_q^-$ except for those edges that intersect the defining arc for the preferred regions of $S^2 \sim P_{\tilde{\phi}}(x_q)$; $\partial D_q$ intersects each of these edges exactly twice so that the points of the edges lying on the defining arc lie in $S^2 \sim D_q$.

Next we choose disjoint arcs $\alpha_q$ for $s < q \leqslant n$ with the following properties: the endpoints of $\alpha_q$ are $x_q^+$ and $x_q^-$; $\alpha_q \cap D_r = \varnothing$ for $1 \leqslant r \leqslant s$ and $s < q \leqslant n$; $\alpha_q$ intersects each edge of $P_{\tilde{\phi}}$ at most once. Now we let $D_q$ be a regular neighborhood of $\alpha_q$ for $s < q \leqslant n$. It is easy to see that $D_q \cap P_{\tilde{\phi}}$ yields a faithful embedding of the syllable graph of the $x_q$ syllables of $\tilde{\phi}$.

In case $s < q \leqslant n$, an arc in $P_{\tilde{\phi}} \cap D_q$ that does not have an endpoint in $D_q$ will be thought of as an $x_q$ syllable of length zero. If we replace these syllable graphs by towns that contain tracks connecting the ends of the syllables we get a reduced RR-system with the desired properties. (See [7, Theorem 4.1].) This completes the proof ot the Fundamental Theorem.

We close this section with some consequences of the above proof. Clearly if $s < q \leqslant n$ then every syllable in $\tilde{\phi}$ with base $x_q$ has exponent $\pm 1$. Suppose $x_q$ is the name of the $q$th town in the RR-system $R$ constructed above (with $s < q \leqslant n$). In order to retrieve $\tilde{\phi}$ from $\Phi_R$ we must choose $m_q = \pm 1 = -p_q$ so that $m_q + p_q = 0$. Each track in station $m_q + p_q$ is to be interpreted to yield a syllable of length zero, that is an arc in the disk $D_q$ separating $x_q^+$ from $x_q^-$ and connecting the endpoints of this track.

## 7. Closed manifolds from RR-systems.

DEFINITION 7.1. Let $P_\phi$ be a faithfully embedded $P$-graph. Suppose $L_1$ and $L_2$ are components of $S^2 \sim P_\phi$ with the property that for some generator $x_i$ of $\phi$ we have $L_1$ lies (clockwise) between $x_{i,j}$ and $C(x_{i,j})$ while $L_2$ lies (counterclockwise) between $B(x_{i,j})$ and $B(C(x_{i,j})) = C^{-1}B(x_{i,j})$. Then $L_1$ and $L_2$ are called *P-contiguous regions*. The equivalence classes of $P$-contiguous regions of $S^2 \sim P_\phi$ are called *P-components*, and if $S^2 \sim P_\phi$ has but one $P$-component we say that it is *P-connected.*

LEMMA 7.2. *If $P_\phi$ is a faithfully embedded P-graph with corresponding manifold $M$, then $S^2 \sim P_\phi$ is P-connected if and only if $\partial M$ is connected.*

PROOF. As we have remarked in §5, we may think of $M$ as having been constructed by attaching 1-handles to a 3-ball in whose boundary $P_\phi$ is embedded (call the resulting handlebody $H$), then attaching 2-handles to $H$ along simple closed curves in $\partial H$ determined by $P_\phi$. The 2-handles attached to $H$ will yield a manifold with connected boundary if and only if the union of the simple closed curves along which the 2-handles are attached does not separate $\partial H$. We start in some component of $S^2 \sim P_\phi$ and follow over a handle without crossing any of the simple closed curves. For notational purposes, let us call the handle in question the $x_0$ handle corresponding to the generator $x_0$. Let us also assume that we crossed the handle going in the direction from $x_0^-$ to $x_0^+$. This path leads us to a new region which is $P$-contiguous with the old region. The lemma follows.

DEFINITION 7.3. $K$ will be called a spine of a closed manifold $M$ if $M - K$ is an open 3-ball.

If $R$ is a RR-system and $\phi \in \Omega_R$, we will give a property of $R$ that is necessary and sufficient for $K_\phi$ to be a spine of a closed orientable 3-manifold.

DEFINITION 7.4. Let $R$ be a RR-system in $S^2$ with towns $D_1, D_2, \ldots, D_n$. Two components $L_1$ and $L_2$ of $S^2 - (R \cup (\bigcup D_i))$ will be called *R-contiguous* if for one of the towns, say $D_{i_0}$, there is an arc $t \subset S^2$ such that $t$ can be added as an additional track to $D_{i_0}$ to yield a new town and one of the endpoints of $t$ lies in $\partial L_1$ while the other lies in $\partial L_2$. $R$-contiguity generates an equivalence relation on the components of $S^2 \sim (R \cup (\bigcup D_i))$. The equivalence classes of $S^2 \sim (R \cup (\bigcup D_i))$ will be called *R-components* and $R$ is said to be *R-connected* if there is only one such equivalence class.

Let $R$ be a given RR-system and let $P$ be a faithfully embedded $P$-graph that is obtained from $R$ by replacing the towns with faithfully embedded syllable graphs. We may expand one of these syllable graphs by adding a new syllable so that the expanded syllable graph is faithfully embedded in $S^2$. The new syllable has its end and beginning lying in the boundaries of two components $L_1$ and $L_2$ of $S^2 \sim R \cup (\bigcup D_i)$. In this situation it is easy to see that $L_1$ and $L_2$ are in the same $P$-component. We have

LEMMA 7.5. *Two components $L_1$ and $L_2$ of $S^2 - (R \cup (\bigcup_{i=1}^{n} D_i))$ are P-contiguous (for any P-graph obtained from R) if and only if they are R-continuous.*

COROLLARY 7.6. *R is R-connected if and only if every faithfully embedded P-graph that originates from R is P-connected.*

THEOREM 7.7. *Let R be a RR-system in $S^2$ with abstract presentation $\Phi_R$. Let $\phi$ be a presentation which originates from $\Phi_R$. If $\Phi_R$ has the same number of abstract relators as generators and R is R-connected then $K_\phi$ is a spine of a closed 3-manifold. Conversely if $K_\phi$ is a spine of a closed 3-manifold with induced embedded P-graph $P_\phi$ then the corresponding RR-system R is R-connected and $\phi$ has the same number of generators as relators. (Cf.* [5].)

PROOF. Since $\phi$ originates from $R$, we have an induced faithful embedding of $P_\phi$ in $S^2$ determining a manifold $M$. To show that $P_\phi$ (as embedded) is a spine of a closed 3-manifold we show that the boundary of $M$ is a 2-sphere. As before we may construct $M$ by attaching 2-handles to a handlebody as determined by the relators of $\phi$. If $\phi$ has $n$ generators then the beginning handlebody has $n$ handles. The Euler characteristic of the boundary of this handlebody is $-2(n-1)$. If we attach a 2-handle to this handlebody the boundary is altered by the deletion of an annulus and the addition of a pair of disks. Thus the Euler characteristic of the new boundary is $-2(n-2)$. After attaching all $n$ of the 2-handles we arrive at a 3-manifold whose boundary has Euler characteristic 2. If one shows that this boundary is connected, it follows that it must be a 2-sphere. This connectivity follows immediately from Lemma 7.2 and Corollary 7.6.

8. **Applications.** We consider the RR-system shown in Figure 8.1. The corresponding abstract presentation is $\langle a, b, c | a^p b^q c^s,\ a^p b^n c^r,\ a^m b^q c^r \rangle$. Clearly all closed manifolds determined by this RR-system have Heegaard decompositions of genus three. We shall show that every manifold in this family has a Heegaard decomposition of genus two. Let us now think of $m, p, n, q, r$, and $s$ as having been chosen subject to the requirements $(m, p) = (n, q) = (r, s) = 1$. Using Theorem 2.6 we can eliminate $a^p$ from the second relator yielding the abstract presentation $\langle a, b, c | a^p b^q c^s, b^{n-q} c^{r-s}, a^m b^q c^r \rangle$. In the RR-system determining the abstract presentation the town labelled $a$ is shown in Figure 8.2. It is easy to see that RR-multiplications can be performed yielding new presentations of the form $\langle a, b, c | a^{\pm p \pm m} W_1, b^{n-q} c^{r-s}, a^m b^q c^r \rangle$ where $W_1$ is an abstract word on $b$ and $c$. In this new RR-system the $a$-town again has the appearance shown in Figure 8.2. Successively performing multiplications we can get words in which any exponent of

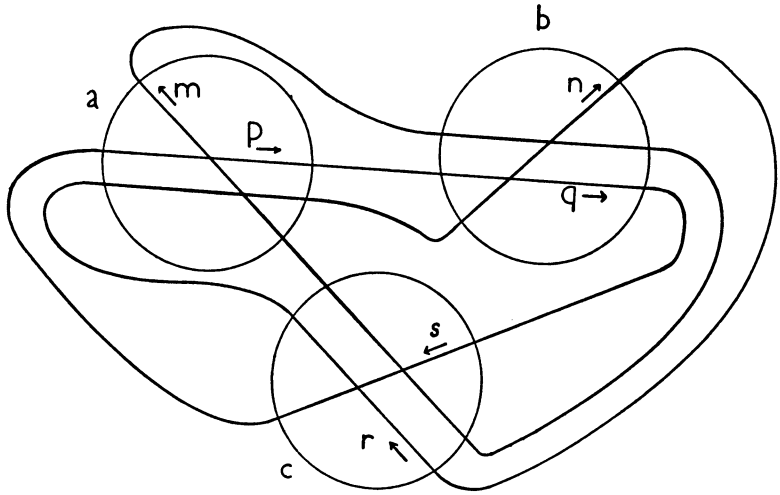



FIGURE 8.1

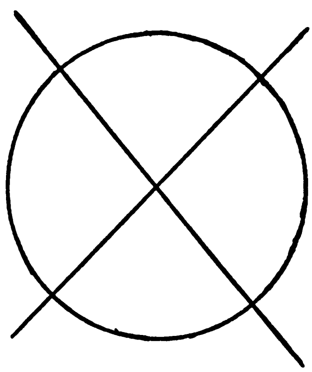


FIGURE 8.2

$a$ of the form $jm + kp$ can be obtained. Since $m$ and $p$ are relatively prime we see that successive multiplication will yield a defining relator for $a$. Upon eliminating $a$ we get a two generator presentation that yields a spine of the same manifold.

In fact, the above argument did not depend on the particular RR-system chosen; it depended only on having $a$ appear in two syllables $a^m$ and $a^p$ in relators $W_1$ and $W_2$ such that $W_1$ contains $a^m$ and no other $a$-syllable and $w_2$ contains the syllable $a^p$ with all other $a$-syllables in $w_2$ being of the form $a^m$. We have proved

THEOREM 8.1. *Let $\phi$ be a presentation that originates from the RR-system R determining an orientable compact 3-manifold $M^3$. Suppose $\phi$ has relators $W_1$ and $W_2$ and a generator $x_1$ with the following property: $w_1$ contains $x_1^{m_1}$ and no other $x_1$-syllable, $w_2$ contains the syllable $x_1^{p_1}$ and all other $x_1$-syllables in $W_2$*

*have exponent* $m_1$. *Then M has a spine corresponding to an* $(n-1)$*-generator group.*

The above theorem can be stated geometrically as a statement about Heegaard decompositions.

As a very special case of this theorem we have

COROLLARY 8.2. *Let M be a closed orientable* 3*-manifold having a spine with corresponding presentation* $\langle a, b | a^m b^n, a^p b^q \rangle$. *Then M is a lens space.*

This corollary was proved in [8]. The special case when $mq - np = \pm 1$ was mentioned in [1] and [11].

J. Birman [2] has given a procedure for enumerating the closed orientable 3-manifolds of Heegaard genus two. This enumeration gives presentations for the fundamental groups of these manifolds (with repetitions). Our results yield a procedure for enumerating all compact orientable 3-manifolds (with duplications) by way of enumerating the presentations (with embeddings) which determine them. Furthermore a large number of repetitions are avoided by using only reduced RR-systems.

Zeeman [10] has shown that the dunce hat cannot be a spine of a counterexample for the Poincaré conjecture. Using techniques developed here, it is easy to show that a much wider class of contractible 2-complexes do not give rise to a counterexample for the Poincaré conjecture. For example one easily shows that $\langle a, b | a^2b^3a^3b^3,\ a^2b^3a^2b^2 \rangle$ determines a contractible 2-complex which can only be a spine of $S^3$. In [3] an algorithm is given for determining whether a 3-manifold of Heegaard genus two is $S^3$. Using two town RR-systems a check can be made to see if one can find a presentation of the trivial group arising from some RR-system. If such a presentation does not reduce to the obviously trivial presentation by allowable operations on the $P$-graph then one has an exciting example to investigate via the algorithm in [3]. In fact the authors strongly suspect that the operations of handledragging and multiplication always allow us to identify a genus two Heegaard decomposition of $S^3$. The unresolved problem in the search is identifying presentations of the trivial group. This identification has been done for two generator presentations with no more than eleven syllables and will be presented in the fourth paper of this series.

Given arbitrary irreducible closed 3-manifolds one might hope to classify most of them by their homotopy properties as suggested by Waldhausen [9]. However a brief examination of manifolds obtained from RR-systems proves discouraging to this hope. In a statistical sense, most closed 3-manifolds have not been shown to be sufficiently large; thus Waldhausen's results cannot be applied.

We close with a conjecture which, if correct, would furnish an algorithmic solution of the homeomorphism problem for compact 3-manifolds.

CONJECTURE. Let $P_{\phi_1}$ and $P_{\phi_2}$ be faithfully embedded $P$-graphs, both determining the manifold $M^3$. Then there is a finite sequence of multiplications, subdivisions and eliminations which transform $P_{\phi_1}$ into $P_{\phi_2}$. Furthermore a bound for the length of this sequence can be determined algorithmically from $\phi_1$ and $\phi_2$.

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